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On the dual König property of the order-interval hypergraph of two classes of N-free posets

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Abstract: Let P be a finite N-free poset. We consider the hypergraph $\mathcal{H}(P)$ whose vertices are the elements of P and whose edges are the maximal intervals of P. We study the dual König property of $\mathcal{H}(P)$ in two subclasses of N-free class.

Keywords: Poset, interval, N-free, hypergraph, König property, dual König property

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1 Introduction

Let $(P, \leq)$ be a finite partially ordered set (briefly *poset* P). A subset of X is called a *chain* (resp. *antichain*) if every two elements in X are comparable (resp. incomparable). The number of elements in a chain is the *length* of the chain. The *height* of an element $x \in P$, denoted by *h(x)*, is the length of a longest chain in P having x as its maximum element. The *height* of a poset P, denoted *h(P)*, is the length of a longest chain in P. The *i-level* or *height-i-set* of P, denoted by N_i, is the set of all elements of P which have height i.

Let p and q be two elements of P. We say *q covers p* and we denote $p \prec q$, if $p \prec v \leq q$ implies $v = q$. Furthermore we denote by $MaxP$ (resp. $MinP$) the set of all maximal (resp. minimal) elements of P. A subset I of P of the form $I = \{v \in P, p \leq v \leq q\}$ (denoted $[p, q]$) is called an *interval*. It is maximal if p (resp. q) is a minimal (resp. maximal) element of P. Denote by $\mathcal{I}(P)$ the family of maximal intervals of P. The hypergraph $\mathcal{H}(P)$=(P,$\mathcal{I}(P)$) whose vertices are the elements of P and whose edges are the maximal intervals of P is said to be the *order-interval hypergraph of P*.

A subset A (resp. T) of P is called *independent* (resp. a *point cover* or *transversal set*) if every edge of $\mathcal{H}$ contains at most one point of A (resp. at least one point of T). A subset $\mathcal{M}$ (resp. $\mathcal{R}$) of $\mathcal{I}$ is called a *matching* (resp. an *edge cover*) if every point of P is contained in at most one member of $\mathcal{M}$ (resp. at least one member of member of $\mathcal{R}$). Let

$$\begin{aligned} \alpha(\mathcal{H}) &= \max\{|A| : A \text{ is independent}\}, \\ \tau(\mathcal{H})) &= \min\{|T| : T \text{ is a point cover}\}, \\ \nu(\mathcal{H}) &= \max\{|\mathcal{M}| : \mathcal{M} \text{ is a matching}\}, \\ \rho(\mathcal{H}) &= \min\{|\mathcal{R}| : \mathcal{R} \text{ is an edge cover}\}. \end{aligned}$$

These numbers are called the *independence number*, the *point covering number*, the *matching number*, and the *edge covering number* of $\mathcal{H}(P)$, respectively. It is easy to see that $\nu(\mathcal{H}) \leq \tau(\mathcal{H})$ and $\alpha(\mathcal{H}) \leq \rho(\mathcal{H})$. We say that $\mathcal{H}$ has the *König property* if $\nu(\mathcal{M}) = \tau(\mathcal{M})$ and *dual König property* if $\nu(\mathcal{H}^*) = \tau(\mathcal{H}^*)$, i.e., $\alpha(\mathcal{H}) = \rho(\mathcal{H})$ since $\alpha(\mathcal{H}) = \nu(\mathcal{H}^*)$ and $\rho(\mathcal{H}) = \tau(\mathcal{H}^*)$. This class of hypergraphs has been studied intensively in the past and we find interesting results from an algorithmic point of view as well as min-max relations [2]-[7] and [9].

Let $P_1 = (E_1, \leq_1)$ and $P_2 = (E_2, \leq_2)$ be two posets such that E_1 and E_2 are disjoint. The *disjoint sum* $P_1 + P_2$ of P_1 and P_2 is the poset defined on $E_1 \cup E_2$ such that $x \leq y$ in $P_1 + P_2$ if and only if $(x, y \in P_1$ and $x \leq_1 y)$ or $(x, y \in P_2$ and $x \leq_2 y$). The *linear sum* $P_1 \oplus P_2$ of P_1 and P_2 is the poset defined on $E_1 \cup E_2$ such that $x \leq y$ in $P_1 \oplus P_2$ if and only if $(x, y \in P_1$ and $x \leq_1 y$) or $(x, y \in P_2$ and $x \leq_2 y$) or $(x \in P_1$ and $y \in P_2)$.

Let $A \subseteq MaxP_1$ and $B \subseteq MinP_2$ with A and B are not empty. The *quasi-series composition* of P_1 and P_2 denoted $P = (P_1, A) * (P_2, B)$ is the poset $P = (E_1 \cup E_2, \leq)$ such that: $x \leq y$ if ($x, y \in E_1$ and $x \leq_1 y$) or ($x, y \in E_2$ and $x \leq_2 y$) or ($x \in E_1$, $y \in E_2$ and there exist $\alpha \in A$, $\beta \in B$ such that $x \leq_1 \alpha$ and $\beta \leq_2 y$.

2 N-free poset

A poset P is said to be *series-parallel* poset, if it can be constructed from singletons P_0 (P_0 is the poset having only one element) using only two operations: disjoint sum and linear sum. It may be characterized by the fact that it does not contain the poset N as an induced subposet [14], [15]. P is called N-*free* if and only if its Hasse diagram does not contain four vertices v_1, v_2, v_3, v_4, where $v_1 \prec v_2$, $v_2 \succ v_3$ and $v_3 \prec v_4$, and v_1 and v_4, v_1 and v_3, v_2 and v_4, are incomparable. The class of N-free posets contains the class of series-parallel posets. Habib and Jegou in [12] defined the *Quasi- Series-Parallel* (QSP for short) class of posets, as the smallest class of posets that contains P_0 and closed under quasi-series composition and lineair sum. They proved that a poset is N-free if and only if it is QSP poset. The following theorem gives many other characterizations of N-free posets (see [11] , [12] and [13]):

Theorem 1 *The four following properties are equivalent:*

i) P is QSP.

ii) P is an N-free poset.

iii) P is a C.A.C. (Chain-Antichain Complete) order (i.e. every maximal chain intersects each maximal antichains).

iv) The Hasse diagram of P is a line-digraph.

v) For every two elements $p, q \in P$, if $N(p) \cap N(q) \neq \emptyset$ then $N(p) = N(q)$, where $N(p)$ denoted the set of all elements of P which cover p in P.

It is known that the order-interval hypergraph $\mathcal{H}(P)$ has the König and dual König properties for the class of series-parallel posets [5]. In [6], it is proved that $\mathcal{H}(P)$ has again the dual König property for the class of a posets that contains the series-parallel posets and whose members have comparability graphs which are distance-hereditary graphs or generalizations of them. If P is an N-free poset, the König property is not satisfied in

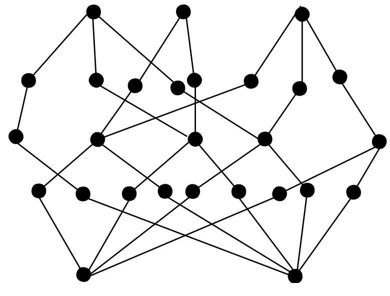


Figure 1: $\nu(\mathcal{H}(P)) = 1$ and $\tau(\mathcal{H}(P)) = 2$

general see [6]. The poset of Figure 1 is an example where $\nu(\mathcal{H}(P)) = 1$, $\tau(\mathcal{H}(P)) = 2$. In this paper, we consider two classes of N-free posets and prove that the dual König property of the order-interval hypergraph of these classes of posets are satisfied.

2.1 Blocks in an N-free poset

There is a useful representation of an N-free poset, namely the *block* (see [1]). If P is an N-free poset with levels $N_1.., N_r$, a block of P is maximal complete bipartite graph in the Hasse diagram of P. More precisely, a block of P is a pair (A_i, B_i), where $A_i, B_i \subset P$ such that A_i is the set of all lower covers of every $x \in B_i$ and B_i is the set of all upper covers of every $y \in A_i$. By convention $(\emptyset, MinP)$ and $(MaxP, \emptyset)$ are blocks

In this paper, we say that (A_i, B_i) and (A_j, B_j) are *adjacent* if there exists at least one vertex of $A_i \cup B_i$ in the same interval in P with at least one vertex of $A_j \cup B_j$. For example, the blocks $(\{b\}, \{c, e\})$ and $(\{a, c\}, \{d\})$ of poset of Figure 2 are adjacent.

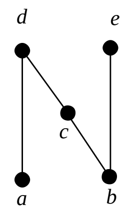


Figure 2: P is N-free with blocks $(\emptyset, \{a, b\}), (\{b\}, \{c, e\}), (\{a, c\}, \{d\})$ and $(\{d, e\}, \emptyset)$.

2.2 N-free poset of Type 1

Definition 1 Let P be a connected poset with levels $N_1, N_2, ... N_r$. We say that P is of *Type 1* if there exists an integer n such that the induced subposet $P_{n,n+1}$ formed from the consecutive levels $N_n \cup N_{n+1}$ is of the form $N_n \oplus N_{n+1}$.

For the class of posets of Type 1, we give the following result:

Theorem 2 *Let P be a poset of Type 1. Then $\mathcal{H}(P)$ has the dual König property and we have:* $\alpha(\mathcal{H}(P)) = \rho(\mathcal{H}(P)) = Max\{|MaxP|, |MinP|\}$.

Proof. We denote by $MinP = \{a_1, a_2, ..., a_k\}$ and $MaxP = \{b_1, b_2, ..., b_l\}$. Consider the family of edges $\mathcal{I}$ of $\mathcal{H}(P)$ such that $\mathcal{I} = \{[a_j, b_j], j = 1, \ldots, k\} \cup \{[a_k, b_j], j = k+1, \ldots, l\}$ if $k \leq l$ and $\mathcal{I} = \{[a_j, b_j], j = 1, \ldots, l\} \cup \{[a_j, b_l], j = l+1, \ldots, k\}$ if $k > l$. It is not difficult to see that $\mathcal{I}$ is an edge-covering family of $\mathcal{H}(P)$ of cardinal equal to $Max\{|MaxP|, |MinP|\}$. Hence, $\alpha(\mathcal{H}(P)) = \rho(\mathcal{H}(P)) = Max\,\{|MaxP|, |MinP|\}$ □

In particular, the order-interval hypergraph of the N-free poset of Type 1 has the dual König property.

3 N-free poset of Type 2

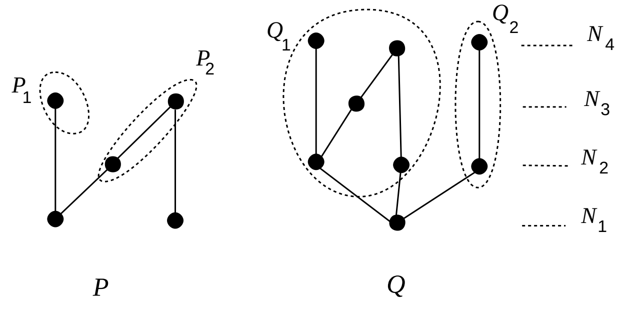
Definitions

1. Let P be a connected poset with levels N_1,N_2,...N_r. We say that P is an N-free poset of *Type 2*, if there exists an integer n such that N_n is the first level where the induced subposet $P_{1,n}$ is disconnected of the form $P_{n,r} = P_1 + P_2 + ... + P_l$, and $\forall i \in L = \{1, .., l\}$, P_i is connected poset of Type 1.

2. We say that the subposet P_i is *linked* with the subposet P_j by a vertex z of N_1, if we can obtain intervals of the form $[z, x]$ and $[z, y]$ for each $x \in MaxP_i$ and $y \in MaxP_j$, and we say z links P_i with P_j.

3. We say that P_i is linked with P_j by the subset R of N_1, if for every element z of R, z links P_i with P_j.

Example 1 The poset P of Figure **??** is N-free of Type 2; it is easy to see that N_2 is the first level where $P_{2,3} = P_1 + P_2$ is disconnected poset with $P_1$and P_2 are of Type 1. On the other hand, Q is an N-free poset but not of Type 2.



In order to prove the dual König property of $\mathcal{H}(P)$, where P is N-free of Type 2, let us introduce the following notations:

Notation

1. For every subposet P_k, we denote by R_k the subset of N_1, where every element of R_k is comparable with all elements of $MaxP_k$, and R_k does not link P_k with any other poset $P_s, s \in L$. The set R_k can be empty.

2. For every subposet P_k, we denote by R'_{ik}, $i \in I_k = \{1, 2, ..., |N_1|\}$, the subset of N_1 which links P_k with the same family of poset $\{P_s\}_{s\in L}$. We can obtain $R'_{ik} = R'_{jl}$ for $i \neq j$ and $k \neq l$.

Observation 3 *The family* $\{R'_{ik}\}_{k\in L, i\in I_k}$ *is pairwise disjoint.*

To illustrate the classe of N-free posets of Type 2, see Figure 3.

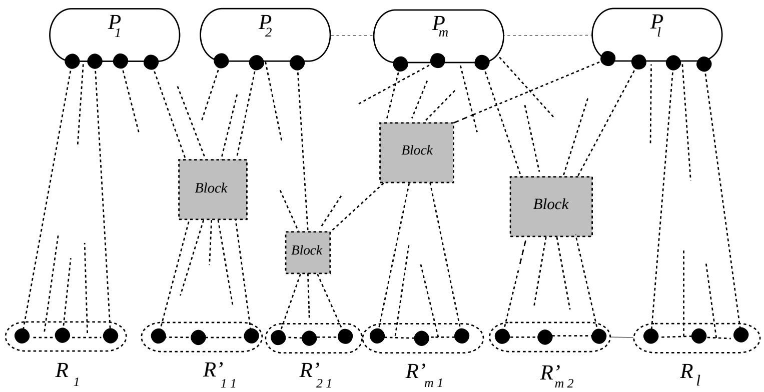


Figure 3: Illustration of an N-free poset of Type 2

3.1 Maximal stable sets of $\mathcal{H}(P)$

In our poset, it is clear that for a linked subposet family $F_k = \{P_l\}_{l\in L}$, we can obtain blocks (A_i, B_i) in the level N_{n-j}, for $j \in \{0, 1, ..., n-1\}$, i.e. B_i intersects N_{n-j}, and every element x of A_i, x links a subfamily F_s of F_k, we say (A_i, B_i) links F_s. Such blocks must exist in N_n since P is N-free poset of Type 2.

We can note the following observation:

Observation 4 *For every block* (A_i, B_i) *which links* F_s*,* B_i *has the following partition:*

$$B_i = \bigcup_{t\in T} B_{i,t}$$

Where $\forall x \in B_{i,t}$*,* x *is comparable with a vertex of* $MinP_t$*, where* $P_t \in F_s$*, and* $|F_s| = |T|$

Let us now give two algorithms to find maximal stable sets of an N-free poset of Type 2, the second algorithm can be applied only after the first.

Maximal Stable-set 1 Algorithm

INPUT: An N-free poset P of Type 2. $F_1, F_2, \ldots, F_m$ all linked subposet families of P.

OUTPUT: Maximal stable set of $\mathcal{H}(P)$.

1. Foreach k, from $k = 1$ to m.
2. Foreach j, from $j = 0$ to $n-1$, in N_{n-j} we determine $C_{k,j}$ by taking for every block (A_i, B_i) which links a subfamily of F_k, one vertex from each $B_{i,t}$ such that:
   i) If there exists a family $\{B_{i,t}\}_i$ from block family which are adjacent pairwise, we take only one vertex from only one set $\{B_{i,t}\}_i$.
   ii) We delete every vertex which is in the same interval with a vertex of $C_{k,t}$, $t < j$.
3. Put $C_k = \bigcup_{j=0}^{n-1} C_{k,j}$.
4. Output $\mathcal{C} = (\bigcup_{k=1}^{m} C_k) \cup (\bigcup_{l \in L} R_l)$. End

Theorem 5 *The set $\mathcal{C}$ is maximal stable set of $\mathcal{H}(P)$.*

Proof. $\mathcal{C}$ is a stable set by construction of every C_k. It remains the maximality of $\mathcal{C}$. We say that an interval I crosses a block (A_i, B_i) if I intersects B_i. Let us show that for every interval I of P, I contains one vertex of $\mathcal{C}$, and this means that for every $x \in P$, $\mathcal{C} \cup \{x\}$ will not be a stable set.
In the case where I does not cross any block, the minimal vertex of I will be in R_l.
Now, In the case where I crosses a block (A_i, B_i), let y be a commun vertex of B_i and I. If $y \in \mathcal{C}$, then I intersect $\mathcal{C}$. Otherwise, $y \notin \mathcal{C}$ that means that y is in the same interval J with an element y' of $\mathcal{C}$. Consequently, I and J will have minimal vertices in R'_{pq} and maximal vertices in $MaxP_l$, this gives $y' \in I$. $\square$

Example 2 The poset of Figure 4 is N-free of Type 2, where P_1, P_2 and P_3 are the supbosets surrounded from left to right, we have: $R'_{11} = R'_{12} = \{a, b\}$, $R'_{21} = R'_{22} = R'_{13} = \{c\}$, $R'_{31} = R'_{23} = \{d\}$, $R'_{41} = R'_{33} = \{e\}$ and $R_3 = \{f\}$. The framed vertices form the maximal stable set $\mathcal{C}$ of $\mathcal{H}(P)$ obtained by *Maximal Stable-set 1* algorithm.

We will need the following definition:

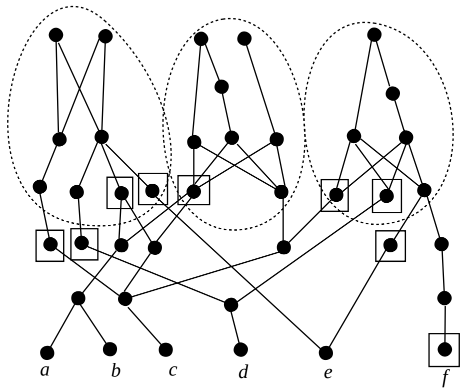


Figure 4: An N-free poset P of Type 2. Applying the *Maximal Stable-Set 1* algorithm on P; the framed vertices form a maximal stable set of $\mathcal{H}(P)$.

Definition 2 In $\mathcal{H}(P)$, for every vertex $x \in P$, the *stable adjacent* M_x to x is the set of all vertices y such that x and y are in the same interval of P, where M_x is stable. M_x can be equal to $\{x\}$. We say M_D is stable adjacent to the set D of P if it is stable adjacent to every vertex of D.

We can write $\mathcal{C} = D_1 \cup D_2 \cup ... \cup D_m$ the stable set obtained from *Maximal Stable-set 1* algorithm, where D_i are subblocks of P. We determine a new maximal stable set $\mathcal{C}'$ from $\mathcal{C}$ as follows:

Maximal Stable-set 2 Algorithm

INPUT : An N-free poset P of Type 2, and maximal stable set $\mathcal{C} = D_1 \cup D_2 \cup ... \cup D_m$.

OUTPUT: A new maximal stable set $\mathcal{C}'$.

1. $\mathcal{C}' := \mathcal{C}$.
2. Foreach i, from $i = 1$ to m.
2. We determine M_{D_i} the stable adjacent to D_i such that $\mathcal{C} - (\cup_{t=1}^{t=i} D_t) \cup (\cup_{t=1}^{t=i} M_{D_t})$ is stable.
3. We take $\mathcal{C}' := \mathcal{C} - (\cup_{t=1}^{t=i} D_t) \cup (\cup_{t=1}^{t=i} M_{D_t})$.
4. Stop.

By construction of $\mathcal{C}'$, we deduce the following result:

Proposition 6 *The set $\mathcal{C}'$ is a maximal stable set of $\mathcal{H}(P)$.*

We denote by C'_k the set of all vertices obtained from every $x_i \in C_k$ using *Maximal Stable-set 2* algorithm.

As a consequence of the previous algorithms, we observe that:

Observation 7 *Consider the subposet family F_k linked by R'_{pq}:*

1. *The set R'_{pq} has the following partition:*

$$R'_{pq} = \bigcup_{s} R'_{pq,s}$$

where for every s, $R'_{pq,s}$ is a stable adjacent to A_s a subset of C'_k.

2. *It will be possible to obtain that the family $\{A_s\}_s$ is pairwise disjoint.*

Proof. To prove the second observation, we suppose that x is a commun vertex of A_s and $A_{s'}$. Let I (resp. J) an interval containing x with minimal element $c_j \in R'_{pq,s}$ (resp. $c_{j'} \in R'_{p'q',s'}$). In I (resp. J) there exists a vertex z (resp. z') wich is incomparable with every vertex of $R'_{p'q',s'}$ (resp. $R'_{pq,s}$) (we take as an example, the vertex z (resp. z') such that $c_j \prec z$ (resp. $c_{j'} \prec z'$)). Otherwise, we will obtain $R'_{pq,s} = R'_{p'q',s'}$ since P is N-free. In this case, we can reconstruct A_s and $A_{s'}$ by starting by z and z' respectively to obtain two new disjoint sets. □

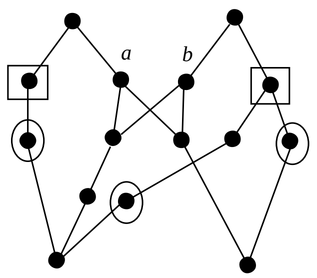


Figure 5: Two different maximal stable sets of $\mathcal{H}(P)$ by applying the *Maximal Stable-set 2* algorithm.

Example 3 The poset of Figure 5 is N-free of Type 2, where $\mathcal{C} = \{a, b\}$. Applying the *Maximal Stable-set 2* algorithm we obtain two different maximal stable sets: $\mathcal{C}'_1$ is the framed vertex set and $\mathcal{C}'_2$ is the surrounded vertex set, we remark that $\mathcal{C}'_2$ verifies the observation 7 (2) while $\mathcal{C}'_1$ does not.

In the remainder of this paper, we suppose that $\mathcal{C}'$ verifies the observation 7 (2).

3.2 Edge covering family of $\mathcal{H}(P)$

In this section, we will present an algorithm to construct an edge covering family of $\mathcal{H}(P)$ where P is an N-free of Type 2.
We denote by $MaxP_l = \{b_1^l, b_2^l, .., b_{|MaxP_l|}^l\}$, $R_l = \{a_1, a_2, .., a_{|R_l|}\}$, $R'_{pq,s} = \{c_1, c_2, .., c_{|R'_{pq,s}|}\}$ and $\bigcup_{i\in I_l} R'_{il} = \{c'_1, c'_2, ..., c'_{m_l}\}$

Theorem 8 *If for every $k \in L$ we have :*

$$|MaxP_k| \geq |R_k| + \sum_{i\in I_k} |R'_{ik}|. \tag{1}$$

Then $\mathcal{H}(P)$ has the dual König property and $\alpha(\mathcal{H}(P)) = \rho(\mathcal{H}(P)) = |MaxP|$.

Proof. For every P_k, we consider the egde family: $\mathcal{I}_k = \{[a_i, b_i], i = 1, .., |R_k|\} \cup \{\left[c'_{j-|R_k|}, b_j\right], j = |R_k|+1, .., |R_k|+m_k\} \cup \{\left[c'_{m_k}, b_s\right], s = m_k+|R_k|+1, .., |MaxP_k|\}$. The union of all $\mathcal{I}_k$, $k \in L$ is an edge covering family of $\mathcal{H}(P)$ with cardinal equals to $|MaxP|$ and as $MaxP$ is a stable set of $\mathcal{H}(P)$ then $\alpha(\mathcal{H}(P)) = \rho(\mathcal{H}(P)) = |MaxP|$. □

We remark that by applying *Maximal Stable-set 2* algorithm to P, we can obtain different maximal stable sets of $\mathcal{H}(P)$ and this depends on the choice of M_{D_i}. In the next algorithm we need to characterize the set $\mathcal{C}'$ as follows:
$\mathcal{C}'$ is determined such that for every subposet family F_k which contains subposets P_l verifying (1), we determine M_{D_i} different to D_i but with the same size, and if $x \in D_i$ is incomparable with all vertices of $MaxP_l$ then M_x will be too. For other subposet families, M_{D_i} does not contain a vertex of $MaxP_m$, where R_m is not empty.

Edge-Cover Algorithm

INPUT: An N-free poset P of Type 2 and the maximal stable set $\mathcal{C}'$.

OUTPUT: An edge covering family $\mathcal{I}(\mathcal{H}(P))$.

Step 1 For every R_l, where P_l does not verify (1), we construct the edge family E_l with $|R_l|$ intervals as follows:

1.1 If $|R_l| \leq |MaxP_l|$: $E_l = \{[a_j, b_j^l], j = 1, 2, ..., |R_l|\}$.

1.2 Otherwise: $E_l = \{[a_j, b_j^l],\ j = 1, 2, ..., |Max.P_l|\} \cup \{[a_t, b_{|MaxP_l|}], t = |MaxP_l| + 1, ..., |R_l|\}$.

Step 2 For every P_l, where P_l verifies (1), we construct the edge family J_l as follows: $J_l = \{[a_i, b_i], i = 1, .., |R_l|\} \cup \{\left[c'_{j-|R_l|}, b_j\right], j = |R_l| + 1, .., |R_l| + m_l\} \cup \{\left[c'_{m_l}, b_s\right],$ $s = m_l + |R_l| + 1, .., |MaxP_l|\}$. We obtain $|MaxP_l|$ intervals.

Step 3 In first, we determine all linked subposet families $F_1, F_2, ..., F_m$. Then, apply this step to $F_k = \{P_l\}_{l \in S_k}$ which is linked by R'_{pq} for $k = 1$ to $k = m$.
In this step, we use the vertices b_t^l of $MaxP_l$, $P_l \in F_k$, which are not used in step 1 or in the application of this step to F_t, where $t < k$; otherwise, we use vertices already used.
Let A'_s be the set A_s deleting all vertices comparable with $MaxP_m$, where P_m verifies (1), and $F'_k = \{P_l\}_{l \in S'_k}$ be the family F_k deleting all subposets verifying (1) . For every $R'_{pq,s}$ we construct the edge family I_s as follows :

3.1 If $|A'_s| \le |R'_{pq,s}|$:
$I_s = \{[c_j, b_t^l], j = 1, 2, ..., |A'_s|\ and\ l \in S'_k\}$. We obtain $|A'_s|$ intervals.

3.2 If $|A'_s| > |R'_{pq,s}|$:
$I_s = \{[c_j, b_t^l], j = 1, 2, ..., |R'_{pq,s}|\ and\ l \in S''_k \subset S_k\} \cup \{[c_1, b_t^l], l \in (S'_k - S''_k)\}$. We obtain $|A'_s|$ intervals.

Step 4 It remains some minimal vertices c_j which are not used in steps 1 and 3 such that $c_j \in R'_{pq,s}$ and R'_{pq} does not link any subposet verifying (1). In this step, we construct J_{c_j} the interval containing c_j and b_t^l a maximal vertex which is not already used, otherwise, J_{c_j} is any interval containing c_j.

Step 5 We take $\mathcal{I}(\mathcal{H}(P))$ the set of all intervals obtained from step 1 to step 4. End.

Theorem 9 *The Edge-Cover algorithm applied to an N-free poset P of Type 2, yields an edge-covering family of $\mathcal{H}(P)$.*

Proof. We can assert that every z of P which is a minimal element, comparable with a vertex of R_m or comparable with a vertex of $MaxP_l$, where P_l verifies (1) is covered by $\mathcal{I}(\mathcal{H}(P))$.
Moreover, if $z > x$, where $x \in A'_s$, then z would be covered by the interval of $\mathcal{I}(\mathcal{H}(P))$ which intersects A'_s.
In other cases, suppose that there exists z of P which is not covered by $\mathcal{I}(\mathcal{H}(P))$, we distiguish two cases.

Case 1. If z is a maximal of P_l and no interval obtained from step 3 or step 4 covers z, then P_l necessarily would verify (1). This contradicts the construction of intervals in these steps.

Case 2. Let $J \notin \mathcal{I}(\mathcal{H}(P))$ containing z and x, where $x \in A'_s$ and $x \not< z$. Let I the interval of $\mathcal{I}(\mathcal{H}(P))$ containing x. The only form of I and J is that they will have maximal elements in $MaxP_l$ and two different minimal elements in $R'_{pq,s}$. z is not covered by I, then for every couple (t, t') of (I, J), where $t \leq x$ and $t' \leq z$, we will have $t \nleq t'$. We supose that a such couple exists.
If t and t' are not in the same interval and $A'_s \cup \{t, t'\} - \{x\}$ is stable, then x can be replaced by t and t' in $\mathcal{C}'$ and this contradicts the construction of $\mathcal{C}'$. Otherwise, we can reconstruct A'_s starting by z, in this case, $R'_{pq,s}$ will be partitionned into at least two subsets, and by applying the Edge-Cover algorithm, z will be covered by the new family.

□

As a consequence of Theorem 9, we have

Corollary 10 *If in the Edge-Cover algorithm, for every vertex x of $MaxP$ (resp. $MinP$), x is taken only once in the construction of $\mathcal{I}(\mathcal{H}(P))$, then P will have the dual König property.*

Proof. In this case, we will have $|\mathcal{I}(\mathcal{H}(P))| = |MaxP|$ (resp. $|MinP|$), and as $MaxP$ and $MinP$ are stable sets of $\mathcal{H}(P)$, therefore $\alpha(\mathcal{H}(P)) = \rho(\mathcal{H}(P)) = |MaxP|$ (resp. $\alpha(\mathcal{H}(P)) = \rho(\mathcal{H}(P)) = |MinP|$). □

Theorem 11 *Let P be an N-free poset of type 2. Then, $\mathcal{H}(P)$ has the dual König property.*

Proof. The main idea of the proof is to use $\mathcal{I}(\mathcal{H}(P))$ obtained from the *Edge-Cover* algorithm for constructing a stable set $\mathcal{C}(\mathcal{H})$ of $\mathcal{H}(P)$ with the same size as $\mathcal{I}(\mathcal{H}(P))$.
Let B_1 (resp. B_2) be the union of all R_l (resp. $MaxP_k$), where P_l (resp. P_k) does not verify (resp. verifies) (1).
From step 1 (resp. step 2) of the Edge-Cover algorithm, B_1 (resp. B_2) is a stable set with the cardinal equals to the cardinal of the union of all E_l (resp. J_l). It becames clear that $B_1 \cup B_2$ is stable set.
The union of all I_s of step 3.1 can be partitioned into 2 subsets, the first denoted by D_1, which is the union of all I_s, where $R'_{pq,s}$ does not link subposets verifying (1), and the second is denoted by D_2.

Let $B_{3,1}$ be the union of all $R'_{pq,s}$, where R'_{pq} does not link subposets verifying (1) and $|R'_{pq,s}| > |A_s|$. $B_{3,1}$ is a stable set with the cardinal equals to $|D_1|$ plus the cardinal of the union of all J_{c_j} of step 4.

We denote by $B_{3,2}$ the union of all A'_s such that $|A'_s| > |R'_{pq,s}|$ or $|A'_s| \leq |R'_{pq,s}|$, where R'_{pq} links subposets verifying (1). From Observation 7 (2), we deduce that there is no common vertex x of A_s and $A_{s'}$ which is covered by two different intervals of $\mathcal{I}(\mathcal{H}(P))$. Consequently, $|B_{3,2}|$ is equal to $|D_2|$ plus the cardinal of the union of all I_s of step 3.2. Consider the following set:

$$\mathcal{C}(\mathcal{H}) = B_1 \cup B_2 \cup B_{3,1} \cup B_{3,2}$$

Hence, it is not difficult to see that $\mathcal{C}(\mathcal{H})$ is a stable set with size $|\mathcal{I}(\mathcal{H}(P))|$. $\square$

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