



An effective approach for integer partitions using exactly two distinct sizes of parts

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Abstract: In this paper we consider the number of partitions of a positive integer n into parts of a specified number of distinct sizes. We give a method for constructing all partitions of n into parts of two sizes, as well as an explicit formula to count them with a new self-contained proof. As a side effect, by using the möbius function we also give a formula for the number of partitions of n into coprime parts.

Keywords: Integer partitions, partitions into parts of different sizes, partitions into parts of two sizes, divisors number, Möbius function.

1 Introduction

A partition of a positive integer n is a sequence of non increasing positive integers n_1 (a_1 times), n_2 (a_2 times), \dots , n_s (a_s times), with $n_i > n_{i+1}$, that sum to n . We sometimes write the such partition $\pi = (n_1^{a_1} n_2^{a_2} \dots n_s^{a_s})$, each n_i is called part of the partition π and a_i its frequency. The partition function $p(n)$ counts the partitions of n . If we ignore some unpublished work of G.W.V. Leibniz, the theory of integer partitions can find its origin in the work of L. Euler [6]. In fact, he made a sustained study of partitions and partition identities, and exploited them to establish a huge number of results in Analysis in 1748. An excellent introduction to this subject can be found in the book of G. E. Andrews [2].

Definition 1 Let $\pi = (n_1^{a_1} n_2^{a_2} \dots n_s^{a_s})$ be a partition of n . We say that π is a partition into k parts with s distinct sizes if

$$\begin{cases} n = a_1 n_1 + \dots + a_s n_s; \\ n_1 > n_2 > \dots > n_s \geq 1; \\ a_1 + \dots + a_s = k; \\ a_1, \dots, a_s \geq 1. \end{cases} \quad (1)$$

Let $t(n, k, s)$ be the number of solutions of system (1) and $t(n, s)$ the total number of partitions of n into s distinct sizes. Then we have

$$t(n, s) = \sum_{k=s}^{\frac{2n-s(s-1)}{2}} t(n, k, s). \quad (2)$$

Example 1 Among 27 partitions of $n = 11$ into 2 distinct sizes, the partitions $(7^1 1^4)$, $(4^2 1^3)$, $(3^1 2^4)$ and $(3^3 1^2)$ are the only ones which are into 5 parts.

This kind of partitions appeared for the first time in the work of P. A. MacMahon [7]. Next, E. Deutsch presented the number of partitions of n into exactly two odd sizes of parts and the number of partitions of n into exactly two sizes of parts, one odd and one even. One can find these values in the Online Encyclopedia of Integer Sequences (OEIS) [8] as A117955 for the first number, A117956 for the second one and A002133 for the number of partitions of n using only 2 types of parts. In the work of Benyahia-Tani and Bouroubi [3], we can find proof of effective and non-effective finiteness theorems on $t(n, k, s)$. We can cite for example the following results:

Theorem 1 For $k \geq s \geq 2$, $n \geq k + \frac{s(s-1)}{2}$ and $n \geq \max\{k, \frac{s(s+1)}{2}\}$, we have

$$t(n, k, s) = \sum_{i=1}^{\lfloor \frac{2n-s(s-1)}{2k} \rfloor} \sum_{j=1}^{k-s+1} t(n - ki, k - j, s - 1), \quad (3)$$

$$t(n, k, 2) = \sum_{i=1}^{\lfloor \frac{n-1}{k} \rfloor} \tau_{k-1\downarrow}(n - ki), \quad (4)$$

where $\tau_{d\downarrow}(k)$ denotes the number of positive divisors of k less than or equal to d .

2 Main Results

One of the aim of this paper is to give an explicit formula for $t(n, k, 2)$ using an effective new approach.

Thus, let consider the system:

$$\begin{cases} n = a_1 n_1 + a_2 n_2; \\ a_1 + a_2 = k; \\ n_1 > n_2 \geq 1; \\ a_1, a_2 \geq 1. \end{cases} \quad (5)$$

and let $m = n_1 - n_2$ throughout the remainder of the paper.

First of all, we introduce the following lemma to prepare the main theorem.

Lemma 2 System (5) has integral solutions if and only if the following conditions are satisfied:

(i) $n \equiv n_2 k \pmod{m}$,

(ii) $\max(1, \lceil \frac{n}{k} \rceil - m + \chi(k|n)) \leq n_2 \leq \lfloor \frac{n}{k} \rfloor - \chi(k|n)$,

where $\chi(k|n) = 1$ if k divides n , and 0 otherwise.

Proof. From system (5), we have

$$\begin{pmatrix} n_1 & n_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} n \\ k \end{pmatrix}.$$

Since $m > 0$, we can write

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{m} \begin{pmatrix} n - n_2k \\ -n + n_1k \end{pmatrix}.$$

Then, system (5) has integral solutions if and only if m divides $n - n_2k$, $n - n_2k > 0$ and $-n + n_1k > 0$. That is,

$$n \equiv n_2k \pmod{m} \quad \text{and} \quad \frac{n}{k} - m < n_2 < \frac{n}{k}.$$

Since k can divide n , and $n_2 \geq 1$, the result holds. ■

From this lemma, we can now derive the following theorem.

Theorem 3 For $k \geq 2$, $n \geq \max\{k, 3\}$, $d = \gcd(n, k)$ and $e|d$, let \mathfrak{J}_e be the set of pairs $(\alpha, \beta) \in \mathbb{N}^2$, such that:

- $1 \leq \alpha \leq \lfloor \frac{n-k}{e} \rfloor$ and $\gcd(\alpha, \frac{k}{e}) = 1$,
- $\beta \equiv \binom{n}{e} \binom{k}{e}^{-1} \pmod{\alpha}$ and $0 \leq \beta \leq \min(\alpha - 1, \lfloor \frac{n}{k} \rfloor - \chi(k|n))$.

Then

$$t(n, k, 2) = \sum_{e|d} \sum_{(\alpha, \beta) \in \mathfrak{J}_e} \left(\left\lfloor \frac{\lfloor \frac{n}{k} \rfloor - \chi(k|n) - \beta}{\alpha} \right\rfloor - \left\lfloor \frac{\max(1, \lfloor \frac{n}{k} \rfloor + \chi(k|n) - \alpha e) - \beta}{\alpha} \right\rfloor + 1 \right).$$

Proof. Put $e = \gcd(m, k)$ and let $\alpha = \frac{m}{e}$, that is $1 \leq \alpha \leq \lfloor \frac{n-k}{e} \rfloor$ and $\gcd(\alpha, \frac{k}{e}) = 1$. By Lemma 2, case (i), we can see that e divides d , and $n_2 \equiv \binom{n}{e} \binom{k}{e}^{-1} \pmod{\alpha}$.

Let $0 \leq \beta < \alpha$, such that $\beta \equiv \binom{n}{e} \binom{k}{e}^{-1} \pmod{\alpha}$. Then

$$n_2 = \beta + t\alpha, \quad t \in \mathbb{Z}.$$

Since $0 \leq \beta < \alpha$ and $\beta \leq n_2 > 0$, then $t \in \mathbb{N}$ and $0 \leq \beta \leq \min(\alpha - 1, \lfloor \frac{n}{k} \rfloor - \chi(k|n))$.

It follows from Lemma 2, case (ii), that

$$\max\left(1, \left\lfloor \frac{n}{k} \right\rfloor + \chi(k|n) - m\right) \leq \beta + t\alpha \leq \left\lfloor \frac{n}{k} \right\rfloor - \chi(k|n).$$

Finally, $t(n, k, 2)$ equals the number of positive integers t , such that

$$\left\lfloor \frac{\max(1, \lfloor \frac{n}{k} \rfloor + \chi(k|n) - m) - \beta}{\alpha} \right\rfloor \leq t \leq \left\lfloor \frac{\lfloor \frac{n}{k} \rfloor - \chi(k|n) - \beta}{\alpha} \right\rfloor.$$

This completes the proof. ■

Remark 4 *One nice application of Theorem 3 concerns the following algorithm which allows us to generate all partitions of n using exactly two distinct sizes of parts.*

Algorithm 1 Partitions into k parts with exactly two distinct sizes of parts

Require: $k \geq 2, n \geq \max\{k, 3\}$

Ensure: Set of quadruple (n_1, a_1, n_2, a_2) ,

$d \leftarrow \gcd(n, k)$

for each divisor e of d **do**

for α **from** 1 **to** $\lfloor \frac{n-k}{e} \rfloor$ **do**

if $\gcd(\alpha, \frac{k}{e}) = 1$ **then**

$\beta \leftarrow \binom{n}{e} \binom{k}{e}^{-1} \pmod{\alpha}$

if $\beta \leq \min(\alpha - 1, \lfloor \frac{n}{k} \rfloor - \chi(k|n))$ **then**

$t_1 \leftarrow \left\lfloor \frac{\max(1, \lfloor \frac{n}{k} \rfloor + \chi(k|n) - \alpha e) - \beta}{\alpha} \right\rfloor$

$t_2 \leftarrow \left\lfloor \frac{\lfloor \frac{n}{k} \rfloor - \chi(k|n) - \beta}{\alpha} \right\rfloor$

for t **from** t_1 **to** t_2 **do**

$n_2 \leftarrow \beta + t\alpha$

$n_1 \leftarrow \alpha e + n_2$

$a_2 \leftarrow \left\lfloor \frac{n - n_1 k}{n_2 - n_1} \right\rfloor$

$a_1 \leftarrow k - a_2$

end for

end if

end if

end for

end for

This algorithm runs in $O(n)$.

Example 2 *Let $n = 11$ and $k = 8$, then $d = \gcd(11, 8) = 1$. So, $e = 1$ is the only one divisor of d . The values of α that satisfies $1 \leq \alpha \leq 3$ and $\gcd(\alpha, 8) = 1$ are 1 or 3.*

1. *For $\alpha = 1$, we get $\beta \equiv 11 \cdot 8^{-1} \pmod{1} = 0$, which is $\leq \min(0, 1)$. The pair $(\alpha, \beta) = (1, 0)$ is then accepted and gives only one value of t :*

$$t = \left\lfloor \frac{1 - 0}{1} \right\rfloor - \left\lfloor \frac{\max(1, 2 - 1) - 0}{1} \right\rfloor + 1 = 1.$$

Therefore, we have only one partition corresponding to the pair $(\alpha, \beta) = (1, 0)$. By applying Algorithm 4, we get:

$$n_2 = 1, n_1 = 2, a_2 = 5 \text{ and } a_1 = 3,$$

and thus the partition $(2^3 1^5)$.

2. For $\alpha = 3$, we get $\beta = 1 \leq \min(2, 1)$, then the pair $(\alpha, \beta) = (3, 1)$ is accepted and gives the values:

$$t = 1, n_2 = 1, n_1 = 4, a_2 = 7 \text{ and } a_1 = 1.$$

Thus the associated partition is $(4^1 1^7)$.

We get, finally

$$t(11, 8, 2) = 2.$$

Example 3 Let $n = 22$ and $k = 8$, then $d = \gcd(22, 8) = 2$. So, we have two divisors of d , $e = 1$ and $e = 2$.

- *Case1: $e = 1$.*

The values of α that satisfies $1 \leq \alpha \leq 14$ and $\gcd(\alpha, 8) = 1$ are 1, 3, 5, 7, 9, 11 or 13.

1. For $\alpha = 1$, we get $\beta = 0$. The pair $(1, 0)$ is accepted and gives the values:

$$t = 1, n_2 = 2, n_1 = 3, a_2 = 2 \text{ and } a_1 = 6,$$

and then the partition $(3^6 2^2)$.

2. For $\alpha = 3$, we get $\beta = 2$. The pair $(3, 2)$ is accepted and gives the values:

$$t = 1, n_2 = 2, n_1 = 5, a_2 = 6 \text{ and } a_1 = 2,$$

and then the partition $(5^2 2^6)$.

3. For $\alpha = 5$, we get $\beta = 4 > \min(4, 2)$, then the pair $(5, 4)$ is rejected.

4. For $\alpha = 7$, we get $\beta = 1$. The pair $(7, 1)$ is accepted and gives the values:

$$t = 1, n_2 = 1, n_1 = 8, a_2 = 6 \text{ and } a_1 = 2,$$

and then the partition $(8^2 1^6)$.

5. For $\alpha = 9$, we get $\beta = 5 > \min(8, 2)$, then the pair $(9, 5)$ is rejected.

6. For $\alpha = 11$, we get $\beta = 3 > \min(10, 2)$, then the pair $(11, 3)$ is rejected.

7. For $\alpha = 13$, we have $\beta = 6 > \min(13, 2)$, then the pair $(13, 6)$ is rejected.

- *Case2: $e = 2$.*

The values of α that satisfies $1 \leq \alpha \leq 7$ and $\gcd(\alpha, 8) = 1$ are 1, 3, 5 or 7.

1. For $\alpha = 1$, we have $\beta = 0$. The pair $(1, 0)$ is accepted and gives $1 \leq t \leq 2$. Applying Algorithm 4, we obtain two partitions corresponding to the pair $(1, 0)$; the first one is $(3^7 1^1)$ for $t = 1$ and the second one is $(4^3 2^5)$ for $t = 2$.

2. For $\alpha = 3$, we get $\beta = 2$. The pair $(3, 2)$ is accepted and gives the values:

$$t = 1, n_2 = 2, n_1 = 8, a_2 = 7 \text{ and } a_1 = 1,$$

and then the partition $(8^1 2^7)$.

3. For $\alpha = 5$, we get $\beta = 4 > \min(4, 2)$, the pair $(5, 4)$ is rejected.

4. For $\alpha = 7$, we get $\beta = 1$. The pair $(7, 1)$ is accepted and gives the values:

$$t = 1, n_2 = 1, n_1 = 15, a_2 = 7 \text{ and } a_1 = 1,$$

and then the partition $(15^1 1^7)$.

We get, finally

$$t(22, 8, 2) = 7.$$

After having counting the number $t(n, k, s)$, it would be of considerable interest to explore the number of partitions of n into k parts with exactly s distinct coprime sizes, which we denote by $g(n, k, s)$. Thus, let set

$$g(n, s) = \sum_{k=s}^{\frac{2n-s(s-1)}{2}} g(n, k, s). \quad (6)$$

Theorem 5 For $k \geq s \geq 2$ and $n \geq \max\{k, \frac{s(s+1)}{2}\}$, we have

$$g(n, k, s) = \sum_{d|n} \mu\left(\frac{n}{d}\right) t(d, k, s), \quad (7)$$

where $\mu(\cdot)$ denotes Möbius function.

Proof. Let $T(n, k, s)$ be the set of partitions of n into k parts with s distinct sizes and $G(n, k, s)$ the subset of the such partitions but with s distinct coprimes sizes. We notice that, the mapping from the set $T(n, k, s)$ to $\bigcup_{d|n} G(d, k, s)$ defined by:

$$(n_1^{a_1} n_2^{a_2} \cdots n_s^{a_s}) \rightarrow \left(\left(\frac{n_1}{\delta}\right)^{a_1} \left(\frac{n_2}{\delta}\right)^{a_2} \cdots \left(\frac{n_s}{\delta}\right)^{a_s} \right),$$

is a bijection, where $\delta = \gcd(n_1, n_2, \dots, n_s)$.

Consequently, we have

$$t(n, k, s) = \sum_{d|n} g(d, k, s). \quad (8)$$

Hence, the result follows by using the Möbius inversion formula. ■

Remark 6 Since $t(d, k, s) = 0$ if $d < \max\left\{k, \frac{s(s+1)}{2}\right\}$, the summation in (7) can be extended only over all divisors d of n such that $\frac{n}{d} \geq \max\left\{k, \frac{s(s+1)}{2}\right\}$. For example, if we take $n = 22$ and $k = 8$, then

$$g(22, 8, 2) = \mu(2)t(11, 8, 2) + \mu(1)t(22, 8, 2),$$

and, according to Examples 2 and 3, we get $g(22, 8, 2) = 7 - 2 = 5$. These partitions are: $(3^7 1^1), (3^6 2^2), (5^2 2^6), (8^2 1^6)$ and $(15^1 1^7)$.

Using Theorems 5 and 3, we can construct the following table:

$n \backslash k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$g(n, 2)$
3	1																		1
4	1	1																	2
5	2	2	1																5
6	1	1	2	1															5
7	3	3	2	2	1														11
8	2	2	2	2	2	1													11
9	3	3	2	3	2	2	1												16
10	2	2	4	1	3	2	2	1											17
11	5	5	3	4	2	3	2	2	1										27
12	2	2	2	2	3	2	3	2	2	1									21
13	6	6	4	5	2	4	2	3	2	2	1								37
14	3	3	5	3	4	1	4	2	3	2	2	1							33
15	4	4	3	3	4	4	2	4	2	3	2	2	1						38
16	4	4	5	3	4	3	3	2	4	2	3	2	2	1					42
17	8	8	5	7	3	5	3	4	2	4	2	3	2	2	1				59
18	3	3	5	2	5	2	4	2	4	2	4	2	3	2	2	1			46
19	9	9	6	7	3	7	3	4	3	4	2	4	2	3	2	2	1		71
20	4	4	4	4	4	3	6	2	3	3	4	2	4	2	3	2	2	1	57

Table 1: $g(n, k, 2), 2 \leq k < n \leq 20$.

From identity (7) we can see that if $k \geq \lfloor \frac{n}{2} \rfloor$, then $t(n, k, 2) = g(n, k, 2)$. In the present theorem we present this observation in a more explicit form.

Theorem 7 For $n \geq \max\{3, k\}$ and $k \geq \max\{2, \lfloor \frac{n}{2} \rfloor\}$, we have

$$t(n, k, 2) = g(n, k, 2) = \tau(n - k) - \chi(n = 2k),$$

where $\tau(n)$ denotes the number of positive divisors of n and $\chi(n = 2k) = 1$ if $n = 2k$, 0 otherwise.

Proof. Let us first notice that if $k \geq 1 + \max\{2, \lfloor \frac{n}{2} \rfloor\}$, then $k \geq \lceil \frac{n+1}{2} \rceil$, and by Identity (4) the result yields (see [3], Corollary 3). Let now $k = \max\{2, \lfloor \frac{n}{2} \rfloor\}$. Since the result is true for $n = 3$, we can assume $k = \lfloor \frac{n}{2} \rfloor$. Let $\pi = (n_1^{a_1} n_2^{a_2})$ be a partition of n into k parts with two distinct sizes. If n is even, then $n_2 = 1$, else $n > (a_1 + a_2) n_2 = k n_2 \geq 2 \lfloor \frac{n}{2} \rfloor = n$, a contradiction. Hence, $n - k = (n_1 - 1) a_1$, in which case $n_1 - 1$ divides $n - k$. So, for each divisor d of $n - k$, we get $n_1 = d + 1$, $a_1 = \frac{n-k}{d} > 0$ and $a_2 = k - \frac{n-k}{d} > 0$, except for $d = 1$, where $a_2 = k - \frac{n-k}{1} = 0$. Thus, the result yields.

Now, if n is odd, then $n_2 = 1$ or $(n_1, n_2) = (3, 2)$. Indeed, if $(n_2 = 2$ and $n_1 \geq 4)$ or $(n_2 \geq 3)$, then $n > 3a_1 + 2a_2 = 2k + a_1 \geq 2 \lfloor \frac{n}{2} \rfloor + 1 = n$, a contradiction. In case of $n_2 = 1$, by the same argument above, we get for each divisor d of $n - k$, $n_1 = d + 1$, $a_1 = \frac{n-k}{d} > 0$ and $a_2 = k - \frac{n-k}{d} > 0$, except for $d = 1$, where $a_2 = k - \frac{n-k}{1} < 0$, which is completed by the partition $(3^{n-2k} 2^{3k-n})$. This completes the proof. ■

Remark 8 *As shown in the proof above, the $t(n, k, 2)$'s partitions have been generated explicitly.*

Acknowledgements

This research was partly supported by L'IFORCE Laboratory.

References

- [1] T. M. Apostol, *Introduction to Analytic Number Theory*. Springer-Verlag, New York (1976).
- [2] G. E. Andrews and K. Eriksson, *Integer Partitions*. Cambridge University Press, Cambridge (2004).
- [3] N.B. Tani and S. Bouroubi, Enumeration of the partitions of an integer into parts of a specified number of different sizes and especially two sizes, *Journal of Integer Sequences*, 14, Article 11.3.6 (2011).
- [4] A. Charalambos Charalambides, *Enumerative Combinatorics*, Chapman & Hall/CRC (2002).
- [5] L. Comtet, *Advanced Combinatorics*, pp. 133–175. D. Reidel Publishing Company, Dordrecht-Holland, Boston, (1974).
- [6] E. Knobloch, Leibniz on combinatorics, pp. 409–430. *Historia Math*, 1 (1974).

- [7] P. A. MacMahon, Divisors of numbers and their continuations in the theory of partitions, *Proc. London Math. Soc.*, **2**, pp. 75–113 (1919).
- [8] OEIS : The online Encyclopedia of Integer sequences. Published electronically at <http://oeis.org>, (2008).
- [9] I. Pak, Partition bijections, a survey, *Ramanujan Journal*, **12**, 5–75 (2006).
- [10] H. Rademacher, On the partition function $p(n)$, *Proc. London, Math. Soc.*, **43**, 241–254 (1937).
- [11] H. S. Wilf, Lectures on integer partitions, unpublished, <http://cis.upenn.edu/wilf>.