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Integer partitions into Diophantine pairs

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Abstract: Let n, a and b be positive integers. The pair (a, b) is called integer partition of n into Diophantine pair if $n = a+b$, $ab+1$ is a perfect square and $a > b$. In this paper we give, for any positive integer n, a closed formula of the number of integer partitions into Diophantine pairs, denoted by $q_D(n, 2)$.

Keywords: Integer partitions, Diophantine pairs, $\tau(n)$, $q_D(n, 2)$

1 Introduction

Let n be an integer, a partition of n is a non increasing sequence of positive integers $n_1, n_2, \ldots, n_k$ whose sum is n, that is $n = n_1 + \cdots + n_k$, with $n_1 \geq \cdots \geq n_k \geq 1$. Each n_i is called a *part* of the partition. The function $p(n)$ denotes the number of partitions of n.

The study of partitions has fascinated several great mathematicians such as Leibniz, Euler, Legendre, Ramanujan, Hardy, Rademacher, Sylvester, Selberg and Dyson, who are interested in many, many number of partitions that satisfy some conditions, denoted $p(n \mid [\text{condition}])$, such as Eulers's identity:

$$p(n \mid [\text{odd parts}]) = p(n \mid [\text{distinct parts}]).$$

Many other interesting problems in the theory of partitions remain unsolved up till now, for example despite a good deal of effort, is to find a simple criterion for deciding whether $p(n)$ is even or odd. There is a vast literature on integer partitions, for more details see for instance [1], [3], [4], [5], [6], [10], [11], [12] and [13].

Now let's move on to another combinatorial concept. A set $\{a_1, a_2, ..., a_m\}$ of m positive integers is called a Diophantine m-tuple if $a_i a_j + 1$ is a perfect square for all i, j with $1 \leq i < j \leq m$. The set $\{1, 3, 8, 120\}$ was the first example of a Diophantine quadruple found by Fermat. A folklore conjecture says that there does not exist a Diophantine quintuple. Arkin, Hoggatt and Strauss [2] shown in general, that for a given Diophantine triple $\{a, b, c\}$, the set $\{a, b, c, d_+\}$ is always a Diophantine quadruple, where $d_+ = a+b+c+2abc+2rst$, and r, s, t are the positive integers satisfying $ab+1 = r^2, ac+1 = s^2, bc+1 = t^2$. The such Diophantine quadruple is called regular.

Dujella [8] proved that there does not exist a Diophantine sextuple and that there exist only finitely many Diophantine quintuples. For more on Diophantine m-tuples results and its history, see for instance Dujella's webpage [9].

The purpose of this paper is to make a link between integer partitions and Diophantine $m-$tuples, more precisely the main goal of this paper is to find the number of partitions of n into two distinct parts, forming a Diophantine pairs.

Definition 1 *A partition of n into exactly two distinct parts, forming a Diophantine pair, is an integral solution of the following system:*

$$\begin{cases} n = a + b, \\ ab + 1 = s^2, \\ a > b \geq 1. \end{cases} \tag{1}$$

Throughout this paper, $q_D(n, 2)$ denotes the number of solutions of system (1).

Example 1 *Among 49 partitions of* $n = 100$ *into* 2 *distinct parts, the partitions:*

$$\begin{aligned} 100 &= 51 + 49, \\ &= 60 + 40, \\ &= 99 + 1, \end{aligned}$$

are the only ones which form a Diophantine pairs:

$$\begin{aligned} 51.49 + 1 &= 50^2, \\ 60.40 + 1 &= 49^2, \\ 99.1 + 1 &= 10^2. \end{aligned}$$

The organization of this paper is as follows: In Sect. 2 we give an upper bound for $q_D(n, 2)$, the number of integer partitions into Diophantine pairs. In Sect. 3 we collect some results to prepare the main result. Section 4 combines the lemmas from Sect 3 to prove the main theorem. Finally, Sect. 5 gives some effective calculus with some examples.

2 Upper bound for $q_D(n, 2)$

Theorem 1 *For* $n \geq 3$, *we have*

$$q_D(n, 2) \leq \left\lfloor \frac{\sqrt{n^2 + 4}}{2} \right\rfloor - \lceil \sqrt{n} \rceil + 1,$$

where $\lfloor w \rfloor$ *and* $\lceil w \rceil$ *denotes the floor and the ceiling of* w *respectively.*

Proof. From (1), we have,

$$-b^2 + nb + 1 = s^2. \tag{2}$$

Since $1 \leq b < n/2$, and the function $f : t \mapsto -t^2 + nt + 1$ is increasing on the interval $\left[1, \dfrac{n}{2}\right[$, it follows

$$n \leq -b^2 + nb + 1 < \frac{n^2 + 4}{4}.$$

Thus, we have

$$\sqrt{n} \leq s < \frac{\sqrt{n^2 + 4}}{2}.$$

Hence, the result follows. ■

3 Some Lemmas

We now present some preliminary results to prepare the main theorem. Let S_n be the set of all solutions of the following Diophantine equation:

$$(n-2x)^2 + (2y)^2 = n^2 + 4,$$

where $(x, y) \in \mathbb{N}^* \times \mathbb{N}^*$ with $n - 2x > 0\cdot$

Lemma 2 *We have*

$$q_D(n, 2) = card(S_n).$$

Proof. From (2) we get

$$(n-2b)^2 + (2s)^2 = n^2 + 4.$$

This shows that we have a 1-1 correspondence between S_n and the set of all solutions of System (1). ■

Let define $r_2^{\mathbb{N}}(n)$ and $r_2^{\mathbb{Z}}(n)$ to be the number of solutions of the equation $x^2 + y^2 = n$, in $\mathbb{N}^2$ and $\mathbb{Z}^2$ respectively. It is clear that if n is not a perfect square, then we have:

$$r_2^{\mathbb{N}}(n) = \frac{r_2^{\mathbb{Z}}(n)}{4}\cdot \tag{3}$$

In his work on number theory, C. G. J. Jacobi established the following result [7]:

$$r_2^{\mathbb{Z}}(n) = 4(\tau_{1[4]}(n) - \tau_{3[4]}(n)), \tag{4}$$

where,

$$\tau_{1[4]}(n) = \sum_{\substack{d/n \\ d\equiv 1(mod\ 4)}} 1 \quad \text{and} \quad \tau_{3[4]}(n) = \sum_{\substack{d/n \\ d\equiv 3(mod\ 4)}} 1.$$

Since $n^2 + 4$ is never a perfect square, from (3) and (4) we get

$$r_2^{\mathbb{N}}(n^2+4) = \tau_{1[4]}(n^2+4) - \tau_{3[4]}(n^2+4). \tag{5}$$

The next Lemma, shows that $\tau_{3[4]}(n^2+4) = 0$, for $n \geq 1$.

Lemma 3 *All odd divisors of $n^2 + 4$ are congruent to 1 modulo 4, for $n \geq 1$.*

Proof. Let $v_2(n)$ denotes the 2-adic order of n. Then $n^2+4 = 2^{v_2(n^2+4)}M$, with M odd. The result holds if $M = 1$. Suppose $M \geq 3$ and let p be an odd prime divisor of M. Since p and 2 are coprime, it exists $u \in \mathbb{Z}$, such that $2u \equiv 1 \ (mod \ p)$, and so

$$4u^2 \equiv 1 \ (mod \ p). \tag{6}$$

Since $n^2+4 \equiv 0 \ (mod \ p)$, we obtain from (6)

$$(un)^2 \equiv -1 \ (mod \ p).$$

By using the first supplement to quadratic reciprocity, we get finally $p \equiv 1 \ (mod \ 4)$, which completes the proof. ■
The following corollary is direct consequence from (5) and Lemma 3.

Corollary 4 *For any positive integer $n \geq 1$, we have*

$$r_2^{\mathbb{N}}(n^2+4) = \frac{\tau(n^2+4)}{1+v_2(n^2+4)}.$$

4 Main result

We are now ready to formulate the main result as follows:

Theorem 5 *For $n \geq 1$, we have*

$$q_D(n,2) = \frac{2\tau(n^2+4)}{3+(-1)^{n+1}+2v_2(n^2+4)} - 1.$$

Proof. Let $T_n = \{(a,b) \in \mathbb{N}^2 : \ a^2+b^2 = n^2+4\}$. We distingue two cases: **Case 1.** If n is odd, let

$$T_n^1 = \{(a,b) \in \mathbb{N}^2 : \ a^2+b^2 = n^2+4, \ a \text{ odd and } b \text{ even}\},$$

$$T_n^2 = \{(a,b) \in \mathbb{N}^2 : \ a^2+b^2 = n^2+4, \ a \text{ even and } b \text{ odd}\}.$$

From corollary 4, we have

$$card(T_n) = \tau(n^2+4). \tag{7}$$

It is clear that $T_n^1 \cap T_n^2 = \emptyset$ and $T_n = T_n^1 \cup T_n^2$. Then,

$$card(T_n) = 2 \ card(T_n^1). \tag{8}$$

Notice that $(x,y) \in S_n$ if and only if $(x,y) \in T_n^1 \setminus \{(n,2)\}$. Which implies

$$card(S_n) = card(T_n^1) - 1. \tag{9}$$

It follows from (7), (8) and (9)

$$card(S_n) = \frac{\tau(n^2+4)}{2} - 1.$$

Case 2. If n is even, so let $n = 2m$, S_n becomes the set of all solutions of the following Diophantine equation: $(m-x)^2 + y^2 = m^2 + 1$, where $(x, y) \in \mathbb{N}^* \times \mathbb{N}^*$ with $m - x > 0$. Let then,

$$T_m = \{(a, b) \in \mathbb{N}^2 : \ a^2 + b^2 = m^2 + 1\}.$$

Since $m^2 + 1$ is never a perfect square, we have

$$card(T_m) = \frac{\tau(m^2+1)}{1 + v_2(m^2+1)}.$$

Notice that $(x, y) \in S_n$ if and only if $(x, y) \in T_m \setminus \{(m, 1)\}$. Then,

$$\begin{aligned} card(S_n) &= card(T_m) - 1 \\ &= \frac{\tau(m^2+1)}{1 + v_2(m^2+1)} - 1. \end{aligned}$$

Since $n^2 + 4 = 4(m^2 + 1)$, it exists M a positive odd integer such that:

$$n^2 + 4 = 2^{v_2(n^2+4)} M \quad \text{and} \quad m^2 + 1 = 2^{v_2(m^2+1)} M.$$

Then,

$$\frac{\tau(m^2+1)}{1 + v_2(m^2+1)} = \frac{\tau(n^2+4)}{1 + v_2(n^2+4)}.$$

Thus

$$card(S_n) = \frac{\tau(n^2+4)}{1 + v_2(n^2+4)} - 1.$$

Finally, the Theorem holds by virtue of Lemma 2. ■

Remark 6 *If we note that*

$$v_2\left(n^2+4\right) = \begin{cases} 0 & \text{if} \quad 2 \nmid n, \\ 3 & \text{if} \quad 2 \parallel n, \\ 2 & \text{if} \quad 4 \mid n, \end{cases}$$

Theorem 5 can be reformulated as follows:

Theorem 7 *For $n \geq 1$, we have*

$$q_D(n, 2) = \begin{cases} \dfrac{\tau\left(n^2+4\right)}{2} - 1 & \text{if} \quad 2 \nmid n, \\ \dfrac{\tau\left(n^2+4\right)}{4} - 1 & \text{if} \quad 2 \parallel n, \\ \dfrac{\tau\left(n^2+4\right)}{3} - 1 & \text{if} \quad 4 \mid n, \end{cases}$$

As an immediate consequence of Theorem 7, we obtain the following corollary:

Corollary 8 *For any positive integer $n \geq 1$, we have*

$$\tau\left(n^2+4\right) \equiv 0 \begin{cases} (mod\ 2) & if \quad 2 \nmid n, \\ (mod\ 4) & if \quad 2 \parallel n, \\ (mod\ 3) & if \quad 4 \mid n. \end{cases}$$

5 Effective calculus and some examples

Example 2 *Let $n = 1000$. We have $n^2+4 = 2^2.53^2.89$, so $\tau(n^2+4) = 18$. From Theorem 7, we get*

$$q_D(1000, 2) = 5.$$

The such partitions are:

$$\begin{aligned} 1000 &= 501 + 499, \\ &= 720 + 280, \\ &= 765 + 235, \\ &= 924 + 76, \\ &= 949 + 51, \end{aligned}$$

of course, all of these partitions verify the Diophantine condition:

$$\begin{aligned} 501.499 + 1 &= 500^2, \\ 720.280 + 1 &= 449^2, \\ 765.235 + 1 &= 424^2, \\ 924.76 + 1 &= 265^2, \\ 949.51 + 1 &= 220^2. \end{aligned}$$

Example 3 *Let $n = 2039$, a prime number. Since $n^2+4 = 5^2.166301$, we get $\tau(n^2+4) = 6$. Therefore, from Theorem 7, we obtain*

$$q_D(2039, 2) = 2.$$

The such partitions are:

$$\begin{aligned} 2039 &= 1304 + 735, \\ &= 1632 + 407, \end{aligned}$$

and

$$\begin{aligned} 1304.735 + 1 &= 979^2, \\ 1632.407 + 1 &= 815^2. \end{aligned}$$

By using a computer algebra package, Theorem 7 allows us to obtain $q_D(n, 2)$ for large values of n. The following table is introduced to illustrate a few:

n	1500	2000	2500	3000	3500	4000	4500	5000	5500	10000	20000
$q_D(n,2)$	1	3	3	7	3	3	3	3	7	11	3

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