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Trees with unique minimum glolal offensive alliance sets

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Abstract: Let $G = (V, E)$ be a simple graph. A set $S \subseteq V$ is called a global offensive alliance if S is a dominating set and for every vertex v in $V - S$ at least half of the vertices in the closed neighborhood of v are in S. The global offensive alliance number is the minimum cardinality of a global offensive alliance in G. In this paper, we give a constructive characterization of trees having a unique minimum global offensive alliance.

Keywords: Domination, global offensive alliance.

1 Introduction

Throughout this paper, $G = (V, E)$ denotes a simple graph with vertex-set $V = V(G)$ and edge-set $E = E(G)$. Let G and H be two graphs with two disjoint vertex sets. Their *disjoint union* is denoted by $G \cup H$, the disjoint union of k copies of G is denoted by kG and the disjoint union of a family of graphs $G_1, G_2, \ldots, G_k$ is denoted by $\cup_{i=1}^{k} G_i$. For every vertex $v \in V(G)$, the *open neighborhood* $N_G(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V(G)$, denoted $d_G(v)$, is the size of its open neighborhood. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex.* If v is a support vertex of a tree T, then $L_T(v)$ will denote the set of the leaves attached at v. Let $L(T)$ and $S(T)$ denote the set of leaves and support vertices, respectively, in T, and let $|L(T)| = l(T)$. As usual, the *path* of order n is denoted by P_n, and the *star* of order n by $K_{1,n-1}$. A *double star* $S_{p,q}$ is obtained by attaching p leaves at an endvertex of a path P_2 and q leaves at the second one. A *subdivision* of an edge uv is obtained by introducing a new vertex w and replacing the edge uv with the edges uw and wv. A subdivided star denoted by SS_k is a star $K_{1,k}$ where each edge is subdivided exactly once. A *wounded spider* is a tree obtained from $K_{1,r}$, where $r \geq 1$, by subdividing at most $r - 1$ of its edges. For a vertex v, let $C(v)$ and $D(v)$ denote the set of *children* and *descendants*, respectively, of v in a rooted tree T, and let $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v.

A *dominating set* of a graph G is a set D of vertices such that every vertex in $V - D$ is adjacent to some vertex in D. *The domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. The concept of domination in graphs, with its many variations, is now well studied in graph theory. For more details, see the books of Haynes, Hedetniemi, and Slater [19, 20].

Among the many variations of domination, we mention the concept of alliances in graphs that has been studied in recent years. Several types of alliances in graphs are introduced in [18], including the offensive alliance that we study here. A dominating set D with the property that for every vertex v not in D,

$$|N_G[v] \cap D| \geq |N_G[v] - D| \tag{1}$$

is called *global offensive alliance set of* G and abbreviated *GOA-set* of G. The *global offensive alliance number* $\gamma_o(G)$ is the minimum cardinality among all GOA-sets of G. A GOA-set of G of cardinality $\gamma_o(G)$ is called γ_o-set of G, or $\gamma_o(G)$-set. Several works have been carried out on global offensive alliances in graphs (see, for example, [2, 6], and elsewhere).

Graphs with unique minimum μ-set, where μ is a some graph parameter, is another concept to which much attention was given during the last two decades. For example, graphs with unique minimum γ-set were first studied by Gunther et al. in [13]. Later this problem was studied for various classes of graphs including block graphs [7], cactus graphs [9], some cartesian product graphs [14] and some repeated cartesian products

[15]. Several works on uniqueness related to other graph parameters have been widely studied, such as locating-domination number [1], paired-domination number [3], double domination number [4], roman domination number [5] and total domination number [17]. Further work on this topic can be found in [8, 10, 11, 12, 16, 21, 22, 23]

The aim of this paper is to characterize all trees having unique minimum global offensive alliance set. We denote such trees as *UGOA-trees*.

2 Preliminaries results

We give in this section the following observations. Some results are straightforward and so their proofs are omitted.

Observation 1 *Let T be a tree of order at least three and $u \in S(T)$. Then,*

(i) *there is a $\gamma_o(T)$-set that contains all support vertices of T,*

(ii) *if D is a unique $\gamma_o(T)$-set, then D contains all support vertices but no leaf,*

(iii) *if $l_T(u) \geq 2$, then u belongs to any γ_o-set(T).*

Proof. (i) and (ii) are obvious. If (iii) is not satisfied, then all leaves attached at u would be contained in D, which is a contradiction with the minimality of D. □

Observation 2 *Let T be a tree obtained from a nontrivial tree T' by joining a new vertex v at a support vertex u of T'. Let D and D' be $\gamma_o(T)$-sets of T and T', respectively. Then,*

(i) $|D'| = |D|$,

$ii)$ *$D \cap V(T')$ is a $\gamma_o(T')$-set,*

(iii) *if T is a UGOA-tree such that u is in any $\gamma_o(T')$-set, then T' is a UGOA-tree.*

Proof. According to Observation 1 (iii), u must be in D since $l_T(u) \geq 2$.

$i)$ D is clearly a GOA-set of T', and then $|D'| \leq |D|$. By Observation 1 (i), we can assume that $u \in D'$. Hence, D' can be extended to a GOA-set of T, which leads to $|D| \leq |D'|$. Thus equality holds.

$ii)$ Since $D \cap V(T') = D$ is a GOA-set of T' with cardinality $|D| = |D'|$, we deduce that $D \cap V(T')$ is a $\gamma_o(T')$-set.

iii) Item (i) together with the fact that u belongs to any $\gamma_o(T')$ imply that D' can be extended to a $\gamma_o(T)$-set. Therefore, the uniqueness of D as a $\gamma_o(T)$-set leads to $D' = D$, which means that D' is the unique $\gamma_o(T')$. □

Observation 3 *Let T be a tree obtained from a nontrivial tree T' different from P_2 by joining the center vertex y of the path $P_3 = x\text{-}y\text{-}z$ at a support vertex v of T'. Let D and D' be $\gamma_o(T)$-sets of T and T', respectively such that each of them contains all support vertices. Then,*

(i) $|D'| = |D| - 1$,

(ii) *$D \cap V(T')$ is a $\gamma_o(T')$-set,*

(iii) *if T is a UGOA-tree, then T' is a UGOA-tree.*

Proof. i) Since $y \in D$ and $v \in D \cap D'$, it follows that $D - \{y\}$ is a GOA-set of T' and so $|D'| \leq |D| - 1$. Moreover, since $v \in D'$, D' can be extended to a GOA-set of T by adding y. Then $|D| \leq |D' \cup \{y\}| = |D'| + 1$ and equality holds.

ii) Since $D \cap V(T') = D - \{y\}$ is a GOA-set of T' with cardinality $|D| - 1 = |D'|$, $D \cap V(T')$ is a $\gamma_o(T')$-set.

iii) Let $B = \{y\}$. In view of item (i), D' can be extended to a $\gamma_o(T)$-set by adding the unique vertex of B. This and item (ii) together with the uniqueness of D imply that $D' = D \cap V(T')$ is the unique γ_o-set of T'. □

Observation 4 *Let k be a positive integer and let T be a tree obtained from a nontrivial tree T' by adding kP_2 joining k pairwise non-adjacent vertices of kP_2 to the same leaf v of T'. Let w be the support vertex adjacent to v, and let D and D' be $\gamma_o(T)$-sets of T and T', respectively. If $w \in D \cap D'$, then the following three properties are satisfies.*

(i) $|D'| = |D| - k$,

(ii) *$D \cap V(T')$ is a $\gamma_o(T')$-set,*

(iii) *if T is a UGOA-tree, then T' is a UGOA-tree.*

Proof. Let $V(kP_2) = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k\}$ and $E(kP_2) = \{x_i y_i : i = 1, 2, \ldots, k\}$. Let v be a leaf of T' and w be the support vertex adjacent to v. We assume that for each $i \in \{1, \ldots, k\}$, y_i is adjacent to v in T.

i) Obviously, all vertices of $\cup_{j=1}^{k} \{y_j\}$ are support vertices in T. Hence, in view of Observation 1 (i), we can assume that D contains all vertices of $\cup_{j=1}^{k} \{y_j\}$. Therefore, since $w \in D$, $D - (\cup_{j=1}^{k} \{y_j\})$ is a GOA-set of T', which means that $|D'| \leq \left|D - (\cup_{j=1}^{k} \{y_j\})\right| = |D| - k$.

Observe that since $w \in D'$, D' can be extended to a GOA-set of T by adding all vertices of $\cup_{j=1}^{k}\{y_j\}$. Hence $|D| \leq \left|D' \cup (\cup_{j=1}^{k}\{y_j\})\right| = |D'| + k$ and so equality holds.

ii) The proof is similar to that of Observation 3(*ii*), by taking $D \cap V(T') = D - (\cup_{j=1}^{k}\{y_j\})$.

iii) The proof is similar to that of (*iii*) of Observation 3(*iii*), by taking $B = \cup_{j=1}^{p}\{y_j\}$. □

Observation 5 *Let $V(T')$ be the vertex-set of a nontrivial tree T', and let D' be a $\gamma_o(T')$-set such $V(T') - D^{'}$ has a vertex w with degree $q \geq 2$ and $|N_{T'}(w) \cap (V(T') - D')| \leq 1$. Let p be a positive integer such that*

$$\begin{cases} p \leq q - 1 \text{ if } |N_{T'}(w) \cap (V(T') - D')| = 0, \\ or \\ p \leq q - 3 \text{ if } |N_{T'}(w) \cap (V(T') - D')| = 1. \end{cases} \quad (2)$$

Let T be a tree obtained from T' by adding p subdivided stars $SS_{k_1}, \ldots, SS_{k_p}$ ($k_i \geq 2$ for all i) with centers $x_1, x_2, \ldots, x_p$, respectively, and joining each x_i ($1 \leq i \leq p$) at w. Let D be a γ_o-set of T. If w and $x_1, x_2, \ldots, x_p$ are not in D, then the following three properties are satisfied.

(*i*) $|D'| = |D| - \sum_{i=1}^{p} k_i$,

(*ii*) *$D \cap V(T')$ is a $\gamma_o(T')$-set,*

(*iii*) *if T is a UGOA-tree, then T' is also a UGOA-tree.*

Proof. For $i \in \{1, \ldots, p\}$, let $S(SS_{k_i})$ be a support vertex-set of SS_{k_i}.

i) Since w together with $x_1, x_2, \ldots, x_p$ are not in D, all vertices of $\cup_{i=1}^{p} S(SS_{k_i})$ must be in D. Therefore, $D \backslash \bigcup_{i=1}^{p} S(SS_{k_i})$ is a GOA-set of T', giving that $|D'| \leq |D| - \sum_{i=1}^{p} k_i$.
On the other hand, let $A = \cup_{i=1}^{p} S(SS_{k_i}) \cup D'$. We have to show that A is a GOA-set of T. For this, it suffices to show that $|N_T[z] \cap A| \geq |N_T[z] - A|$ for each $z \in \{w, x_1, x_2, \ldots, x_p\}$. Indeed, we have to distinguish between two cases.
Case 1. $z = x_i$, for some $i \in \{1, \ldots, p\}$.
We have then

$$|N_T[z] \cap A| = |N_T[z] \cap \cup_{i=1}^{p} S(SS_{k_i})| = k_i \geq 2,$$

and

$$|N_T[z] - A| = |\{z, w\}| = 2.$$

Case 2. $z = w$.
We have then

$$|N_T[z] \cap A| = \begin{cases} q & \text{if } |N_{T'}(w) \cap (V(T') - D')| = 0, \\ q - 1 & \text{if } |N_{T'}(w) \cap (V(T') - D')| = 1. \end{cases}$$

and

$$|N_T[z] - A| = \begin{cases} p+1 & \text{if } |N_{T'}(w) \cap (V(T') - D')| = 0, \\ p+2 & \text{if } |N_{T'}(w) \cap (V(T') - D')| = 1. \end{cases}$$

According to (2), we have in each case $|N_T[z] \cap A| \geq |N_T[z] - A|$ for each $z \in \{w, x_1, x_2, \ldots, x_p\}$. Therefore A is a GOA-set of T, giving that $|D| \leq |A| = |D'| + \sum_{i=1}^{p} k_i$. Hence the equality holds.

ii) Using the fact that $D \cap V(T') = D \backslash \cup_{i=1}^{p} S(SS_{k_i})$, this property follows in a similar manner as the proof of Observation 3(*ii*).

(*iii*) This property follows in a similar manner as the proof of Observation 3(*iii*), by taking $B = \cup_{i=1}^{p} S(SS_{k_i})$. □

3 The main result

In order to characterize the trees with unique minimum global offensive alliance, we define a family $\mathcal{F}$ of all trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_r$ $(r \geq 1)$ of trees, where T_1 is the path P_3 centered at a vertex y, $T = T_r$, and if $r \geq 2$, T_{i+1} is obtained recursively fom T_i by one of the following operations. Let $A(T_1) = \{y\}$.

- Operation $\mathcal{O}_1$: Attach a vertex by joining it to any support vertex of T_i. Let $A(T_{i+1}) = A(T_i)$.
- Operation $\mathcal{O}_2$: Attach a path $P_3 = u\text{-}v\text{-}w$ by joining v to any support vertex of T_i. Let $A(T_{i+1}) = A(T_i) \cup \{v\}$.
- Operation $\mathcal{O}_3$: Let w be a support vertex of T_i that satisfies one of the following two conditions.
  1. $l_{T_i}(w) \geq 3$,
  2. $|N_{T_i}[w] \cap A(T_i)| < |N_{T_i}(w) \cap (V(T_i) - A(T_i)|$ or
     - ∗ either $l_{T_i}(w) = 2$ and $N_{T_i}(w) - A(T_i)$ has a vertex w_t such that $|N_{T_i}(w_t) \cap A(T_i)| \leq |N_{T_i}[w_t] \cap (V(T_i) - A(T_i)| + 1$,
     - ∗ or $l_{T_i}(w) = 1$ and $N_{T_i}(w) - A(T_i)$ has two vertices w_p, w_q so that for $l = p, q$, $|N_{T_i}(w_l) \cap A(T_i)| \leq |N_{T_i}[w_l] \cap (V(T_i) - A(T_i)| + 1$.

Let kP_2 be the disjoint union of $k \geq 1$ copies of P_2, and let B be a set of k pairwise non-adjacent vertices of kP_2. Add kP_2 and attach all vertices of B to a same leaf in T_i that is adjacent to w. Let $A(T_{i+1}) = A(T_i) \cup B$.

- Operation $\mathcal{O}_4$: Let $w \in V(T_i) - A(T_i)$ be a vertex of degree $q \geq 2$ in T_i such that $|N_{T_i}(w) \cap (V(T_i) - A(T_i))| \leq 1$. Attach $p \geq 1$ subdivided stars SS_{k_i} ($k_i \geq 2$ for $1 \leq i \leq p$) with support vertex-set $S(SS_{k_i})$ and of center x_i by joining x_i to w for all i such that

$$p \leq \begin{cases} q-1 & \text{if } |N_{T_i}(w) \cap (V(T_i) - A(T_i))| = 0, \\ q-3 & \text{if } |N_{T_i}(w) \cap (V(T_i) - A(T_i))| = 1. \end{cases}$$

Let $A(T_{i+1}) = A(T_i) \cup (\cup_{i=1}^{p} S(SS_{k_i}))$.

Before stating our main result, we need the following lemma.

Lemma 6 *If $T \in \mathcal{F}$, then $A(T)$ is the unique $\gamma_o(T)$-set.*

Proof. Let $T \in \mathcal{F}$. We proceed by induction on the number of operations, say r, required to construct T. The property is true if T is a path P_3 centered at y since $A(T) = \{y\}$ is the unique $\gamma_o(T)$-set. This establishes the base case.
Assume that for any tree $T' \in \mathcal{F}$ that can be constructed with $r-1$ operations, $A(T')$ is the unique $\gamma_o(T')$-set. Let $T = T_r$ with $r \geq 2$ and $T' = T_{r-1}$. We distinguish between four cases.

Case 1. T is obtained from T' by using Operation $\mathcal{O}_1$.
Assume that T is obtained from T' by attaching an extra vertex at a support vertex u of T'. In view of Observation 1 (ii), $u \in A(T')$. Hence $A(T')$ can be extended to a GOA-set of T. By Observation 2 (i), $\gamma_o(T) = \gamma_o(T')$, implying that $A(T')$ is a $\gamma_o(T)$-set. Applying the inductive hypothesis to T', $A(T')$ is the unique $\gamma_o(T')$-set. It follows that $A(T) = A(T')$ is the unique $\gamma_o(T)$-set.

Case 2. T is obtained from T' by using Operation $\mathcal{O}_2$.
$A(T') \cup \{v\}$ is a GOA-set of T. By Observation 3 (i), $\gamma_o(T) = \gamma_o(T') + 1$, meaning that $A(T') \cup \{v\}$ is a $\gamma_o(T)$-set. The inductive hypothesis sets that $A(T')$ is the unique $\gamma_o(T')$-set. Thus $A(T) = A(T') \cup \{v\}$ is the unique $\gamma_o(T)$-set.

Case 3. T is obtained from T' by using Operation $\mathcal{O}_3$.
$A(T') \cup B$ is a GOA-set of T. Observation 4 (i) sets that $\gamma_o(T) = \gamma_o(T') + k$, which means that $A(T') \cup B$ is a $\gamma_o(T)$-set. By the inductive hypothesis, $A(T')$ is the unique $\gamma_o(T')$-set. Thus $A(T) = A(T') \cup B$ is the unique $\gamma_o(T)$-set.

Case 4. T is obtained from T' by using Operation $\mathcal{O}_4$.
$A(T') \cup (\cup_{i=1}^{p} S(SS_{k_i}))$ is a GOA-set of T. According to Observation 5 (i), we have $\gamma_o(T) = \gamma_o(T') + \sum_{i=1}^{p} k_i$, whence, $A(T') \cup (\cup_{i=1}^{p} S(SS_{k_i}))$ is a $\gamma_o(T)$-set. By the inductive hypothesis, $A(T')$ is the unique $\gamma_o(T')$-set. It follows that $A(T) = A(T') \cup (\cup_{i=1}^{p} S(SS_{k_i}))$ is the unique $\gamma_o(T)$-set. □

Remark that in each case, $A(T_{i+1})$ is obtained from $A(T_i)$ by adding all support vertices in $T_{i+1} \backslash T_i$. Hence the following corollary is immediate.

Corollary 7 *Let $T \in \mathcal{F}$ and $S(T)$ be a set of support vertices in T. Then $\gamma_o(T) \geqslant |S(T)|$.*

Now we are ready to prove our main result.

Theorem 8 *A tree T is a UGOA-tree if and only if $T = K_1$ or $T \in \mathcal{F}$.*

Proof. It is obvious that $T = K_1$ is a UGOA-tree. Also, Lemma 6 states that any member of $\mathcal{F}$ is a UGOA-tree. Now, we prove the converse by induction on the number n of vertices of T. The converse holds trivially for $n = 1$ and 3 but not for $n = 2$ since P_2 is not a UGOA-tree. When $n = 4$, T is either a $K_{1,3}$ or a P_4. Clearly P_4 is not a UGOA-tree, whilst $K_{1,3}$ is a UGOA-tree that can be obtained from a P_3 using operation $\mathcal{O}_1$, and so $K_{1,3} \in \mathcal{F}$. If $n = 5$, then T is either a double star $S_{1,2}$ which is not a UGOA-tree, or it is a $K_{1,4}$ or P_5 that are UGOA-tree since $K_{1,4}$ can be obtained from $K_{1,3}$ by using operation $\mathcal{O}_1$, and P_5 can be obtained from a P_3 by using operation $\mathcal{O}_3$. Therefore $K_{1,4}$ and P_5 are in $\mathcal{F}$. This establishes the base case.

Now, let $n \geq 6$ and assume that any tree T' of order $3 \leq n' < n$ with the unique $\gamma_o(T')$-set is in $\mathcal{F}$. Let T be a tree of order n with the unique $\gamma_o(T)$-set D and let $s \in S(T)$. By Observation 1 (ii), $s \in D$. If $l_T(s) \geq 3$, then let T' be the tree obtained from T by removing a leaf adjacent to s and let D' be a $\gamma_o(T')$-set. Then, clearly $n' = |V(T')| = n - 1 \geq 5$, and $l_{T'}(s) \geq 2$, so $s \in D'$ by Observation 1 (iii). According to Observation 2 (ii), T' is UGOA-tree. Applying the inductive hypothesis to T', we get $T' \in \mathcal{F}$. Thus T is obtained from T' by operation $\mathcal{O}_1$, implying that $T \in \mathcal{F}$. Assume now that

$$\text{for each } x \in S(T),\ l_T(x) \leq 2. \tag{3}$$

Root T at a vertex r of maximum eccentricity. Let u be a support vertex of maximum distance from r and let u' be a leaf adjacent to u. Let v and w be the parents of u and v, respectively, in the rooted tree. We consider two cases.

Case 1. $v \in D$.
If $l_T(u) = 1$, then $D \cup \{u'\} - \{u\}$ is a $\gamma_o(T)$-set, contradicting the uniqueness of D as a $\gamma_o(T)$-set. Hence by (3), $l_T(u) = 2$. We claim that $v \in S(T)$. Suppose not. Then either $w \in D$ and so $D - \{v\}$ is a GOA-set of T with cardinality less than $|D|$, contradicting the minimality of D, or $w \notin D$ and so $D - \{v\} \cup \{w\}$ is a $\gamma_o(T)$-set, contradicting the uniqueness of D as a $\gamma_o(T)$-set. This completes the proof of the claim. Let $T' = T - T_u$ and D' be a γ_o-set of T'. By Observation 1(i), we can assume that D' contains all support vertices in T'. Since $|V(T_u)| = 3$, it follows that $n' = |V(T')| = n - 3 \geq 3$ and so $T' \neq P_2$. By Observation 3(iii), T' is a UGOA-tree. Applying our inductive hypothesis, we get $T' \in \mathcal{F}$. Thus, T can be obtained from T' by operation $\mathcal{O}_2$ and so $T \in \mathcal{F}$.

Case 2. $v \notin D$.
According to Observation 1(ii), $v \notin S(T)$ and so $l_T(v) = 0$. Let $k = |N_T(v) - \{w\}|$. We have then $d_T(v) = k + 1$ and since $u \in N_T(v) - \{w\}$, we clearly deduce $k \geq 1$. For $i \in \{1, \ldots, k\}$, let $u_i \in N_T(v) - \{w\}$ such that $u_1 = u$. The choice of v sets that

$$u_i \in S(T),\ l_T(u_i) \geq 1 \text{ and so } u_i \in D \text{ for all } i. \tag{4}$$

Hence by (3), we have $1 \leq l_T(u_i) \leq 2$ for all i. Assume first that $l_T(u_j) = 2$ for some j in $\{1, \ldots, k\}$. Without loss of generality, let $j = 1$. Then u has a further neighbor $u'' \neq u'$ in T. Let $T' = T - \{u''\}$ and D' be any γ_o-set of T'. Clearly u' is the unique leaf of u in T'. We claim that $u \in D'$. Suppose not. Then u' and v must be in D' and therefore $D'' = (D' \backslash \{u'\}) \cup \{u\}$ is a further $\gamma_o(T)$-set other than D (since v belongs to D'' and not to D), a contradiction. This completes the proof of the claim. We have $n' = n - 1 \geq 5$. By Observation 2(iii), T' is a UGOA-tree. Applying our inductive hypothesis to T', we get $T' \in \mathcal{F}$. Hence T is obtained from T' by operation $\mathcal{O}_1$, implying that $T \in \mathcal{F}$. Assume now that

$$l_T(u_i) = 1 \text{ and hence } d_T(u_i) = 2 \text{ for all } i. \tag{5}$$

For all $i \in \{1, \ldots, k\}$, let u_i' be the unique leaf adjacent to u_i (with $u_1' = u'$). We distinguish between two subcases, depending on whether w belongs to D or not.

Case 2.1. $w \in D$.
In view of (5), $T_v - \{v\} = kP_2$ with $V(kP_2) = \{u_1, u_2, \ldots, u_k, u_1', u_2', \ldots, u_k'\}$ and $E(kP_2) = \{u_i u_i' : i = 1, 2, \ldots, k\}$. Let $T' = T - (T_v - \{v\})$. Clearly $v \in L(T')$ and $w \in S(T')$. If $n' = |V(T')| = 2$, then T is a wounded spider with exactly one non-subdivided edge and in this case, it is not difficult to see that such a graph is not a UGOA-tree. Hence assume that $n' \geq 3$. We claim the following:
If $l_T(w) \in \{0, 1\}$, then one of the two conditions holds:

$C_1 :$ $|N_T[w] \cap D| \leq |N_T(w) \cap (V(T) - D|$.

$C_2 :$ (i) either $l_T(w) = 1$ and $N_T(w) - D$ has a vertex w_t such that

$$|N_T(w_t) \cap D| \leq |N_T[w_t] \cap (V(T) - D)| + 1$$

(ii) or, $l_T(w) = 0$ and $N_T(w) - D$ has two vertices w_p, w_q such that for $l \in \{p, q\}$,

$$|N_T(w_l) \cap D| \leq |N_T[w_l] \cap (V(T) - D)| + 1.$$

Indeed, suppose that C_1 and C_2 are not satisfied. Assume first that $l_T(w) = 1$, so $L_T(w)$ has exactly one vertex, say w'. In this case $D - \{w\} \cup \{w'\}$ is a $\gamma_o(T)$-set different from D, a contradiction. Now, assume that $l_T(w) = 0$. Since C_2 is not fulfilled, item (ii) of C_2 is satisfied for at most one vertex in $N_T(w) - D$, say w''. Then $D - \{w\} \cup \{w''\}$ is a $\gamma_o(T)$-set different from D, a contradiction. If no vertex in $N_T(w) - D$ for which item (ii) of C_2 is satisfied, then $D - \{w\} \cup \{v\}$ is a $\gamma_o(T)$-set different from D, which leads to a contradiction again. This complete the proof of the claim.
Observe that when $l_{T'}(w) \in \{1, 2\}$, the previous claim remain true by replacing D by D' and T by T'. Thus, according to Observation 4 (iii), T' is a UGOA-tree. By induction on T', we get $T' \in \mathcal{F}$. Since T is obtained from T' by using operation $\mathcal{O}_3$, we directly obtain $T \in \mathcal{F}$.

Case 2.2. $w \notin D$.
By Observation 1(ii), $w \notin S(T)$ and so $l_T(w) = 0$. Since v and w are in $V(T) - D$, v must

have at least two neighbors in D. Hence $d_T(v) = k+1 \geq 3$. Let t be the parent of w, and let X, Y and Z be the following sets

$$Y = C(w) \cap S(T),\ X = C(w) - Y \text{ and } Z = D(w) \cap (S(T) - Y).$$

Observe that $v \in X$, $u \in Z$, $N_T(w) = \{t\} \cup X \cup Y$ and every vertex in Z plays the same role as u. Therefore by (4), we have $Z \subset D$ since $Z \subset S(T)$, and by (5), every vertex in Z has exactly two neighbors such that one of them is a leaf and the other one is in X. Furthermore, as $v \in X$, $u_i \in Z$ for all $i \in \{1, \ldots, k\}$, so $|Z| \geq k \geq 2$. Notice also that $|X| \geq 1$ since $v \in X$. Likewise $|Y| \geq 1$ since D is a $\gamma_o(T)$-set. It is clear that $Y \subseteq S(T)$ and thus $Y \subseteq D$ by Observation 1(ii). Setting

$$X = \{x_1, x_2, \ldots x_p\} (p \geq 1) \text{ with } x_1 = v \text{ and } |Y| = q - 1\ (q \geq 2).$$

Since every vertex in X plays the same role as v, $x_i \in V(T) - D$ for all $i \in \{1, \ldots, p\}$. Setting

$$p_i = |N_T(x_i) - \{w\}| \text{ for } i = 1, \ldots, p.$$

Then $p_1 = k$. Since for all $i \in \{1, \ldots, p\}$, x_i and w are in $V(T) - D$, x_i must have at least two neighbors in Z. Hence $d_T(x_i) = p_i + 1 \geq 3$. This means that for all $i \in \{1, \ldots, p\}$, $V(T_{x_i})$ induces a subdivided star SS_{p_i} of order p_i+1 centered at x_i. Since $w \in V(T) - D$, inequality (1) is valid by replacing v with w. This gives

$$p \leq q - 1 \text{ if } t \in D, \text{ or } p \leq q - 3 \text{ otherwise.} \tag{6}$$

Let $T' = T - (\cup_{i=1}^{p} T_{x_i})$ and D' be a $\gamma_o(T')$-set. Observe that T' contains at least one P_3 as an induced subgraph, which means that $n' = |V(T')| \geq 3$. For all $i \in \{1, \ldots, p\}$, let $S(SS_{p_i})$ be the support vertex-set of SS_{p_i}. Clearly $\cup_{i=1}^{p} S(SS_{p_i}) = Z$ and $N_{T'}(w) = Y \cup \{t\}$, so

$$d_{T'}(w) = q \geq 2.$$

According to Observation 1 (i), we can assume that $Y \subset D'$ since $Y \subset S(T')$. Then t is the only neighbor of w in T' that may not be in D', that is

$$|N_{T'}(w) \cap (V(T') - D')| \leq 1.$$

If $t \in D'$, then the minimality of D' sets that $w \in V(T') - D'$, because otherwise, we replace w by t in D'.
By Observation 5 (ii) and (iii), we have $D' = D \cap V(T')$. Hence $t \in D$ if and only if $t \in D'$. Notice that if $t \in D'$, then $N_{T'}(w) \cap (V(T') - D')$ is an empty-set, otherwise, t would be the unique vertex of $N_{T'}(w) \cap (V(T') - D')$. Thus (6) can be rewritten as follows.

$$\text{If } |N_{T'}(w) \cap (V(T') - D')| = 0, \text{ then } p \leq q - 1,$$

and

$$\text{if } |N_{T'}(w) \cap (V(T') - D')| = 1, \text{ then } p \leq q - 3.$$

Again Observation 5(iii) sets that T' is a UGOA-tree. Applying the inductive hypothesis to T', we deduce $T' \in \mathcal{F}$. Now since T can be obtained from T' by operation $\mathcal{O}_4$, and finally $T \in \mathcal{F}$. This completes the proof of Theorem 8. □

4 Open Problems

The previous results motivate the following problems.

1- Characterize other UGOA-graphs.

2- Characterize trees with unique minimum defensive alliance sets (UGDA).

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